MOTION OF A PAIR OF BUBBLES IN A LIQUID OF LOW VISCOSITY

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 3, pp. 29–36, 1968

Lagrange's equations are used to examine the long-range interaction of bubbles. The Lagrange function equals the kinetic energy of an ideal liquid flowing around a bubble. The generalized external forces include the upthrust and the viscous resistance to flow around each buble. The azimuthal angle is increased by the long-range interaction. The locus for the relative motion is calculated for: 1) the case in which the relative speed is fairly high, which allows one to neglect the effects of viscosity on the collision time, 2) low relative speed, where the viscous forces determine the motion. Estimates are given for the differential effective cross-section for elastic scattering and the coalescence cross-section.

1. Lagrange equation. Consider a system of two spherical bubbles whose radii are $a_i (i = 1, 2)$. The bubbles move with speeds u_i in a liquid whose kinematic viscosity is ν. It is assumed that the Reynolds number $R_i = u_i a_i/\nu$ satisfies $1 \ll R_i < 300$.

We assume that the velocity distribution is as for an ideal liquid, in which case we can derive the Laplace equation for the velocity potential apart from the region $r_i' < a_i (r_i' = |\mathbf{r}_i'|,\ \mathbf{r}_i' = \mathbf{r} - \mathbf{r}_i)$:

$$\Delta\Phi = 0, \tag{1.1}$$

in which $\mathbf{r}_i$ are the coordinates of the center of the i-th bubble. The boundary conditions at $r_i' = a_i$ take the form

$$n_i\left(\frac{\partial\Phi}{\partial r^\alpha} - u_i^\alpha\right) = 0, \qquad n_i^\alpha = \frac{r_i'^\alpha}{r_i'}, \tag{1.2}$$

and also

$$\Phi \to 0 \quad \text{for} \quad r \to \infty. \tag{1.3}$$

Summation with respect to repeated subscripts is understood. Up to terms of order $(a/r)^3$ inclusive (a is the mean radius of a bubble and r is the distance between the centers), we get the following expression [1] for the potential:

$$\Phi = -\sum_{i=1}^{2} \frac{a_i^3}{2}\varphi_i^\alpha u_i^\alpha, \qquad \varphi_i^\alpha = \frac{r_i'^\alpha}{r_i'^3} - \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{2} a_j^3 \Lambda^{\alpha\beta}\frac{r_j'^\beta}{r_j'^3},$$

$$r = |\mathbf{r}|,\ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \qquad \Lambda^{\alpha\beta} = \frac{3r^\alpha r^\beta}{r^5} - \frac{\delta^{\alpha\beta}}{r^3},$$

$$\delta^{\alpha\beta} = \begin{cases} 1, & (\alpha = \beta) \\ 0, & (\alpha \neq \beta) \end{cases}. \tag{1.4}$$

The Lagrange function for the bubbles in a liquid of low viscosity is the kinetic energy of an ideal liquid of viscosity ρ flowing around the system of bubbles, and this to terms of order $(a/r)^3$ is

$$T = 1/3\pi\rho\,(a_1^3 u_1^2 + a_2^3 u_2^2 - 3a_1^3 a_2^3 u_1^\alpha \Lambda^{\alpha\beta} u_2^\beta). \tag{1.5}$$

As previously [2], we derive the following expression for the generalized external forces:

$$Q_i^\alpha = -\frac{4\pi}{3}\rho a_i^3 g^\alpha - 12\pi\mu a_i\Big(u_i^\alpha - \sum_{\substack{j=1\\ j\neq i}}{}' a_j^3 u_j^\beta \Lambda^{\beta\alpha}\Big). \tag{1.6}$$

Then the Lagrange equation can be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{u}_i} - \frac{\partial T}{\partial \mathbf{r}_i} = Q_i \tag{1.7}$$

for the bubbles in a low-viscosity liquid up to terms of order $(a/r)^3$ inclusive:

$$\frac{du_1^\alpha}{dt} = -\frac{3}{2}a_2^3 u_2^\beta \frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}u_2^\gamma -$$

$$- 2\left(g^\alpha + \frac{3}{2}a_2^3 g^\beta \Lambda^{\beta\alpha}\right) - \frac{18\nu}{a_1^2}\left(u_1^\alpha + \frac{1}{2}a_2^3 u_2^\beta \Lambda^{\beta\alpha}\right),$$

$$\frac{du_2^\alpha}{dt} = \frac{3}{2}a_1^3 u_1^\beta \frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}u_1^\gamma -$$

$$- 2\left(g^\alpha + \frac{3}{2}a_1^3 g^\beta \Lambda^{\beta\alpha}\right) - \frac{18\nu}{a_2^2}\left(u_2^\alpha + \frac{1}{2}a_1^3 u_1^\beta \Lambda^{\beta\alpha}\right). \tag{1.8}$$

The right-hand side of (1.8) contains accelerations due to the short-range force F from the hydrodynamic interaction, which decreases as r^{-4}, and also due to long-range forces that decrease as r^{-3}.

We assume u_1 and u_2 to be on the order of $u_0 = ga^2/9\nu$ (the speed of steady-state rise in a bubble of mean radius a), which allows us to estimate the orders of the accelerations due to F and Q:

$$F \sim \left(\frac{ga^2}{9\nu}\right)^2 \frac{a^3}{r^4}, \qquad Q \sim \frac{ga^3}{r^3}. \tag{1.9}$$

Then, if we consider collision in the range

$$r/a \ll (a\sqrt{ga}/9\nu)^2 = R/9. \tag{1.10}$$

we can neglect Q relative to F, and Eqs. (1.8) become

$$\frac{du_1^\alpha}{dt} = -\frac{3}{2}a_2^3 u_2^\beta \frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}u_2^\gamma - 2g^\alpha - \frac{18\nu}{a_1^2}u_1^\alpha,$$

$$\frac{du_2^\alpha}{dt} = \frac{3}{2}a_1^3 u_1^\beta \frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}u_1^\gamma - 2g^\alpha - \frac{18\nu}{a_2^2}u_2^\alpha. \tag{1.11}$$

These equations are also suitable for $r/a \gtrsim R/9$, since in that region the acceleration due to Q is negligible relative to g.

We have shown [2] that the Lagrange equations are applicable to the motion if the acceleration is such that the speed alters little in a time a/u_0. Let b denote the distance of closest approach between the paths of the two particles in the absence of interaction; then the maximum acceleration due to the pair interaction is on the order of $a^3u_0^2/b^4$. The velocity change in a time a/u_0 then satisfies $u_0(a/b)^4 \ll u_0$ if $(a/b)^4 \ll 1$.

Equations (1.11) are applicable to the description of remote pair collisions of bubbles if $(a/b)^4 \ll 1$. if $(a/b)^4 \ll 1$.

2. Two bubbles in an ideal liquid. Equations (1.11) can be replaced by simpler ones if we consider the collision under conditions such that the effective collision time $\tau_0 = b/u_-$ (where u_- is the initial relative velocity) is negligibly small relative to $\tau = a^2/18\nu$ (the time of velocity relaxation in response to viscosity and upthrust). Then the velocity change at a separation $r \approx b$ is determined mainly by the forces present in an ideal liquid in the absence of external forces. The other forces cannot substantially change the speeds in time τ_0.

Then the velocity change in the region $r \approx b$ is described by

$$\frac{du_1^\alpha}{dt} = -\frac{3}{2} a_2^3 u_2^\beta \frac{\partial \Lambda^{\beta\gamma}}{\partial r^\alpha} u_2^\gamma,$$

$$\frac{du_2^\alpha}{dt} = \frac{3}{2} a_1^3 u_1^\beta \frac{\partial \Lambda^{\beta\gamma}}{\partial r^\alpha} u_1^\gamma, \tag{2.1}$$

which are Lagrange's equations for two spheres moving in an ideal liquid in the absence of external forces, the Lagrange function being defined by (1.5).

The Lagrange function is invariant under the parallel displacement $\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{a}$, $\mathbf{a} = \text{const}$, so the total momentum is conserved:

$$P^\alpha = m_1 u_1^\alpha + m_2 u_2^\alpha - {}^3/_2\ (a_1^3 + a_2^3)\ m_1 m_2\ (u_1^\beta + u_2^\beta)\Lambda^{\beta\alpha},$$

$$m_1 = a_1^3 / (a_1^3 + a_2^3),\quad m_2 = a_2^3 / (a_1^3 + a_2^3). \tag{2.2}$$

The vector for the relative velocity is

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2. \tag{2.3}$$

The equation for the change in $\mathbf{u}$ is readily derived from (2.1):

$$\frac{du^\alpha}{dt} = -\frac{3}{2}(a_2^3 u_2^\beta u_2^\gamma + a_1^3 u_1^\beta u_1^\gamma)\frac{\partial \Lambda^{\beta\gamma}}{\partial r^\alpha}. \tag{2.4}$$

If (2.2) and (2.3) are solved for $\mathbf{u}_1$ and $\mathbf{u}_2$, neglecting terms of order $(a/r)^3$, we have

$$\mathbf{u}_1 = \mathbf{P} + m_2\mathbf{u},\qquad \mathbf{u}_2 = \mathbf{P} - m_1\mathbf{u}. \tag{2.5}$$

Upon substitution of (2.5) into the right-hand side of (2.4), we get an equation for the relative motion of the bubbles:

$$\frac{du^\alpha}{dt} = -\frac{3}{2}(a_1^3 + a_2^3)(P^\beta P^\gamma + m_1 m_2 u^\beta u^\gamma)\frac{\partial \Lambda^{\beta\gamma}}{\partial r^\alpha}. \tag{2.6}$$

This equation is solved by successive approximation. As zeroth approximation we take the motion without interaction:

$$\mathbf{r} = \mathbf{b} + \mathbf{u}_- t \tag{2.7}$$

where $\mathbf{b}$ is the vector for the perpendicular velocity u_- of the relative motion before collision, where the length b of the vector equals the distance of closest approach; $\mathbf{b}$ lies in a plane passing through the center of the second bubble and the path of the relative motion of the first bubble in the absence of interaction.

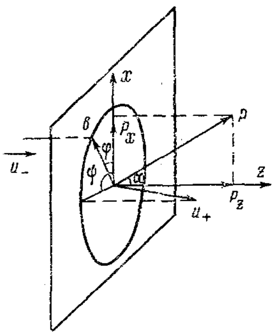

Fig. 1

The first approximation is as follows for the relative velocity u_+ after collision:

$$u_+^\alpha = u_-^\alpha - \frac{3}{2}(a_1^3 + a_2^3)\int_{-\infty}^{+\infty} P^\beta \frac{\partial \Lambda^{\beta\gamma}}{\partial r^\alpha} P^\gamma\, dt. \tag{2.8}$$

The last term in (2.6) is the total derivative with respect to time:

$$u^\beta \frac{\partial \Lambda^{\alpha\gamma}}{\partial r^\alpha} u^\gamma = u^\beta \frac{\partial \Lambda^{\alpha\beta}}{\partial r^\gamma} u^\gamma = \frac{d}{dt} u^\beta \Lambda^{\alpha\beta}, \tag{2.9}$$

and so the integral of (2.9) with respect to time with infinite limits is zero. The integral of (2.8) may conveniently be calculated in the coordinate system of Fig. 1, where the z-axis lies along $\mathbf{u}$, the x-axis is in the plane of $\mathbf{u}_-$ and P and is perpendicular to $\mathbf{u}_-$, and the y-axis is perpendicular to the x- and z-axes. The total momentum $\mathbf{P}$ in this coordinate system has the components

$$P_x = P \sin\alpha,\qquad P_z = P\cos\alpha, \tag{2.10}$$

in which α is the angle between $\mathbf{u}_-$ and $\mathbf{P}$.

The vectors in the plane $z = 0$ are defined by the complex numbers $x + iy$. Then vector $\mathbf{b}$ takes the form

$$\mathbf{b} = be^{i\varphi} = x + iy \tag{2.11}$$

The integral of (2.8) in this coordinate system becomes:

$$\mathbf{u}_+ = \mathbf{u}_- - \frac{3}{2}\frac{a_1^3 + a_2^3}{u_-} P^2 \sin^2\alpha \frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial \mathbf{r}}\frac{1}{r}\,dz. \tag{2.12}$$

It is known from electrostatics that the potential produced by an unbounded charged filament with unit charge per length ($2\ln b = -2\,\mathrm{Re\,Ln}\,b$) is the real part of the analytic function. The integral of (2.12) is the field strength with that potential.

The Cauchy-Riemann conditions give us for an arbitrary analytic function of $b = x + iy$ that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\mathrm{Re}\,f(b) = \overline{\frac{\partial}{\partial x} f(b)} = \overline{\frac{df}{db}}. \tag{2.13}$$

Then (2.13) gives

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial \mathbf{r}}\frac{1}{r}\,dz = -2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\mathrm{Re\,Ln}\,b =$$

$$= -2\overline{\frac{d}{db}\mathrm{Ln}\,b},\quad \frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial \mathbf{r}}\frac{1}{r}\,dz = -2\frac{d^3}{db^3}\mathrm{Ln}\,b.$$

Then the relative velocity is

$$\mathbf{u}_+ = \mathbf{u}_- + 6\frac{P^2}{u_-}\sin^2\alpha\,\frac{a_1^3 + a_2^3}{b^3}e^{3i\varphi}. \tag{2.14}$$

We see from (2.14) that $\mathbf{u}_+$ retains its length $|\mathbf{u}_+| = |\mathbf{u}_-|$ up to terms of order $(a/b)^3$ inclusive and is determined by the angle θ between $\mathbf{u}_+$ and $\mathbf{u}_-$ as well as by the azimuthal angle in the plane $z = 0$. We can express θ and ψ in terms of the collision parameters:

$$\theta = 6(P / u_-)^2 \sin^2\alpha\,(a_1^3 + a_2^3) / b^3,\ \psi = 3\varphi. \tag{2.15}$$

The azimuthal angle alters on account of the noncentral forces during collision. After scattering, particles enter the angular range ψ to $\psi + d_\psi$ from the initial angular range

$${}^1/_3\,\psi,\ {}^1/_3\,\psi + {}^1/_3\,d\psi.$$

and also

$$\psi\,{}^1/_3 \pm {}^2/_3\pi,\ {}^1/_3\,\psi \pm {}^2/_3\,\pi + {}^1/_3\,d\psi.$$

Integration with respect to φ allows one to write the differential scattering cross-section as

$$d\sigma = 2\pi\,({}^4/_3)^{1/3}(a_1{}^3 + a_2{}^3)^{2/3}\frac{P^2\sin^2\alpha}{u_-{}^2}\,\frac{d\theta}{\theta^{5/3}} = \\ = 2\pi\left(\frac{4}{3}\right)^{1/3}(a_1{}^3 + a_2{}^3)^{2/3}\frac{[\mathbf{u}_1\times\mathbf{u}_2]^2}{u_-{}^4}\,\frac{d\theta}{\theta^{5/3}}. \quad (2.16)$$

The collision turns the vector for the relative velocity through θ. The velocity component u_- of the motion of the first bubble along the initial direction alters by

$$m_2u_-\,(1 - \cos\,\theta) = {}^1/_2\,m_2u_-\,\theta^2. \quad (2.17)$$

The kinetic theory of gases [3] shows that the transport collision cross-section

$$\sigma_t = \int(1 - \cos\theta)\,d\sigma \quad (2.18)$$

defines quantities such as the mean velocity change and the mean rate of change of energy, and (2.16) shows that the main contribution to this comes from close collisions. We assume that (2.15) applies as to order of magnitude up to $\theta \approx 1$ to get

$$\sigma_t \sim \pi(a_1 + a_2)^2\ \ [\mathbf{u}_1\times\mathbf{u}_2]^2\,/\,u_-{}^4. \quad (2.19)$$

It is of interest to estimate the maximum possible value of σ_t. Equations (2.1) apply for $b/u_- \ll a^2/18\nu$, i.e., for $u_- \gg 18\nu b/a^2$; $[\mathbf{u}_1\times\mathbf{u}_2]^2 \leqslant u_0^2u_-^2$, so

$$\sigma_t \leqslant {}^1/_{18}\,\pi\,(a_1 + a_2)^2R. \quad (2.20)$$

Then a nearly isotropic distribution of the relative velocities can lead to the transport collisional cross-section; however, $\sigma_t = 0$ for the case of bubbles with parallel initial velocities, as (2.19) shows.

The bubbles may coalesce if b is small enough. The probability of this may be expressed via a coalescence cross section.

Consider the relative motion in an ideal liquid. It follows from (2.6) that the projections of the radius vector r on the plane z = 0 and on the z-axis are

$$x + iy = be^{i\varphi} - \frac{3}{2}\,\frac{a_1{}^3 + a_2{}^3}{u_-{}^2}\times \\ \times\int\limits_{-\infty}^{u_-t}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\int\limits_{-\infty}^{z'}(P^\alpha P^\beta + m_1m_2u_-{}^\alpha u_-{}^\beta)\,\Lambda^{\alpha\beta}dz'\,dz,$$

$$z = u_-t - \frac{3}{2}\,\frac{a_1{}^3 + a_2{}^3}{u_-{}^2}\times \\ \times\int\limits_{-\infty}^{u_-t}(P^\alpha P^\beta + m_1m_2u_-{}^\alpha u_-{}^\beta)\,\Lambda^{\alpha\beta}\,dz. \quad (2.21)$$

We calculate the integrals via (2.13) to get

$$\frac{x + iy}{b} = e^{i\varphi} + \frac{3}{2}\,\frac{a_1{}^3 + a_2{}^3}{b^3}\left\{\frac{P^2}{u_-{}^2}\left[2\left(\frac{r + z}{b} - \frac{b}{r}\right)\sin^2\alpha e^{3i\varphi} - \right.\right. \\ - \left(1 + \frac{z}{r}\right)\sin 2\alpha e^{2i\varphi} - \frac{b^2z}{r^3}\sin 2\alpha\cos\varphi e^{i\varphi} + \\ \left.\left. + \frac{b^3}{r^3}\,(\cos^2\alpha - \sin^2\alpha\cos^2\varphi)\,e^{i\varphi}\right] + m_1m_2\frac{b^3}{r^3}e^{i\varphi}\right\}$$

$$z = u_-t - \frac{3}{2}\,\frac{a_1{}^3 + a_2{}^3}{b^3}\left\{\frac{P^2}{u_-{}^2}\left[\left(1 + \frac{z}{r}\right)\sin^2\alpha\cos 2\varphi - \right.\right. \\ - \frac{b^2z}{r^3}\,(\cos^2\alpha - \sin^2\alpha\cos^2\varphi) - \\ \left.\left. - \frac{b^3}{r^3}\sin 2\alpha\cos\varphi\right] - m_1m_2\frac{b^2z}{r^3}\right\}. \quad (2.22)$$

The condition for (2.22) to be applicable is $(a/b)^3 \times (P/u_-)^2$, which is obeyed since $u_- \gg 18\,\nu\,b/a^2$, $P \sim u_0$ for long-range collisions with $b \gg a(R/18)2/5$. Collisions at shorter distances in the range $a \leqslant b \leqslant a(R/18)2/5$ produce scattering through large angles, while coalescence can occur if $b \leqslant a$.

The locus of (2.22) takes an especially simple form for the collision of bubbles whose initial velocities are parallel ($\alpha = 0$):

$$x + iy = be^{i\varphi}\left(1 + \varepsilon\frac{b^3}{r^3}\right),\quad z = u_-t\left(1 + \varepsilon\frac{b^3}{r^3}\right), \\ \varepsilon = \frac{3}{2}\,\frac{a_1{}^3 + a_2{}^3}{b^3}\left(\frac{P^2}{u_-{}^2} + m_1m_2\right). \quad (2.23)$$

The coalescence cross section for a relative velocity of $u_- \gg 18\nu/a$ with $\alpha = 0$ should be somewhat less than $\pi(a_1 + a_2)^2$, the geometrical collision cross section.

3. Quasi-stationary problem. Consider a collision for which the effective collision time $\tau_0 = b/u$ is large relative to the velocity relaxation time $\tau_i = a_i^2/18\,\nu$ ($i = 1, 2$) for each bubble. Then the acceleration of a bubble is

$$\frac{du_i}{dt} \sim u_i/\tau,$$

which may be considered negligible relative to the viscous friction.

The equation of motion for the bubbles can be written as

$$u_1{}^\alpha = -2\tau_1g^\alpha - \frac{3}{2}\,a_2{}^3\tau_1u_2{}^\beta\,\frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}\,u_2{}^\gamma, \\ u_2{}^\alpha = -2\tau_2g^\alpha + \frac{3}{2}\,a_1{}^3\tau_2u_1{}^\beta\,\frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}\,u_1{}^\gamma. \quad (3.1)$$

The speed of the i-th bubble differs little from $-2\tau_ig^\alpha$, so (3.1) can be replaced by

$$u_1{}^\alpha = -2\tau_1g^\alpha - 6a_2{}^3\tau_1\tau_2{}^2g^\beta\frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}\,g^\gamma \\ u_2{}^\alpha = -2\tau_2g^\alpha + 6a_1{}^3\tau_1{}^2\tau_2g^\beta\frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}\,g^\gamma \quad (3.2)$$

We assume that $|a_1 - a_2| \ll a$; $a \approx a_1$, to get the equation for the relative motion as

$$\frac{dr^\alpha}{dt} = -2\Delta\tau g^\alpha - 12a^3\tau^3g^\beta\frac{\partial\Lambda^{\beta\gamma}}{\partial r^\alpha}\,g^\gamma \\ \Delta\tau = \tau_1 - \tau_2,\qquad \tau = \tau_1 \quad (3.3)$$

Equation (3.3) takes the following form in the polar coordinate system r, θ ($r = |\mathbf{r}|$, θ is the angle between g and r):

$$\frac{dr}{dt} = 2\Delta\tau g\cos\theta + 36a^3\tau^3g^2\,\frac{3\cos^2\theta - 1}{r^4}, \\ r\frac{d\theta}{dt} = -2\Delta\tau g\sin\theta + 36a^3\tau^3g^2\,\frac{\sin 2\theta}{r^4}. \quad (3.4)$$

Then the locus of the relative motion is

$$\frac{d\xi}{\xi d\theta} = \frac{3\cos^2\theta - 1 - \xi^4\cos\theta}{\sin 2\theta - \xi^4 \sin\theta},$$

$$\xi = \frac{r}{a}\left(\frac{4\Delta a}{Ra}\right)^{1/4} \qquad \binom{\Delta a = a_1 - a_2}{R = ga^3/9\nu^2}. \tag{3.5}$$

Integration of (3.5) gives a family of loci in terms of parameter C:

$$\xi^2 \sin^2\theta = C \pm \sqrt{C^2 - \sin^3\theta \sin 2\theta}. \tag{3.6}$$

Figure 2 shows the integral curves in ξ, θ coordinates. Equation (3.5) has a singular point, with the saddle point at

$$\cos\theta_0 = 1/\sqrt{5}, \quad \xi_0^4 = 2/\sqrt{5}.$$

The integral curve of (3.6) passes through that point if

$$C = C_0 = (4/5)^{5/4} \approx 0.76,$$

The motion along the locus is described

$$\xi\frac{d\theta}{d\zeta} = -\sin\theta + \frac{\sin 2\theta}{\xi^4},$$

$$\zeta = \frac{2}{9}\left(\frac{4\Delta a}{Ra}\right)^{1/4}\frac{g\Delta a}{\nu}t. \tag{3.7}$$

We see that the bubbles interact in a region whose dimensions are of order $\xi = 1$, while the duration of the interaction is on the order of $\xi = 1$. However, the interaction time increases substantially if $C - C_0 \ll C_0$, and then the locus near $\theta_0 = \arccos\sqrt{5}/5$ is

$$\xi^2 = 5/4\left[C + \sqrt{C^2 - C_0^2 + 32/25\sqrt{5}\,(\theta - \theta_0)^2}\right]$$

The following is the time taken to traverse the part of the locus from $\theta = \theta_0$ to $\theta_0 - \Delta\theta$:

$$\Delta\zeta = 1/2\,(5/4)^{3/4}\int_0^\eta \frac{dx}{\sqrt{1+5x^2}+x}, \quad \eta = \frac{C_0\Delta\theta}{\sqrt{C^2 - C_0^2}},$$

and this has logarithmic divergence as C approaches C_0.

We thus can assume that the bubbles have an unstable coupled state with a finite lifetime.

Equations (3.2) apply if $b/u_- \gg a^2/18\nu$, i. e., $\Delta a/a$ $\Delta a/a \ll 9b/Ra$ a $r \gg a$; however, if we suppose that these equations are suitable for describing the motion also when $r \approx a$, collision with direct contact occurs if

$$b < b_0 = a\sqrt{C_0}\,(Ra/\Delta a)^{1/4} = 0.88\,a\,(Ra/\Delta a)^{1/4}.$$

The following is the largest cross section leading to fusion:

$$\sigma_c = \pi a^2 C_0\sqrt{Ra/\Delta a}. \tag{3.8}$$

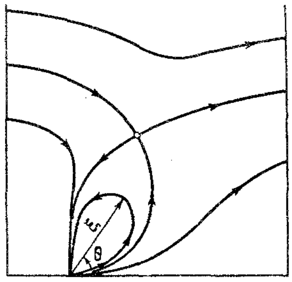

Fig. 2

Now $b < b_0$ for these collisions, so we get as follows from the conditions for (3.2) to apply:

$$\Delta a/a \ll (9/R^2\sqrt{C_0})^{1/5}R.$$

Then in this approximation

$$\sigma_c \gg \pi a^2 (R/3)^{4/5}.$$

The divergence of the cross section for $\Delta a \to 0$ is unimportant in calculating the mean number of collisions in unit time per bubble for a system with an average of N bubbles per unit volume:

$$-\frac{dN}{Ndt} = \frac{1}{2}u_-\sigma_c N = \pi a^2\frac{ga^2}{9\nu}NC_0\left(\frac{R\Delta a}{a}\right)^{1/2}, \tag{3.9}$$

if the mean radius of a bubble is a and the standard deviation Δa satisfies

$$\Delta a/a \ll (9/R^2\sqrt{C_0})^{1/5}R.$$

The rate of coalescence for bubbles similar in radius is then

$$-\frac{dN}{Ndt} \ll 3R^{1/5}\pi a^2\frac{ga^2}{9\nu}N. \tag{3.10}$$

These estimates show that the coalescence probability for bubbles of similar radius is on the same order as the collision probability for bubbles of radius of order a but differing appreciably in size, which approach one another at a speed $u_- \sim ga^2/9\nu$, provided that the coalescence cross section is $\sigma_c \sim \pi a^2$.

We are indebted to V. G. Levich and V. V. Tolmachev for discussions.

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